Characterising Logics through their Admissible Rules

Jeroen Goudsmit
Utrecht University
July 14th 2014, 17:30 – 17:50
A / $\Delta$ admissible
$\sigma A$ is derivable

$A / \Delta$ admissible

$\sigma C$ is derivable for some $C \in \Delta$
\[ \sigma A \text{ is derivable} \]

\[ A \vdash \Delta \text{ admissible} \]

\[ \sigma C \text{ is derivable for some } C \in \Delta \]
\[
\neg C \rightarrow A \lor B \\
(\neg C \rightarrow A) \lor (\neg C \rightarrow B)
\]
\[ A \lor B \]
\[ \{A, B\} \]
Łukasiewicz 1952
Kreisel and Putnam 1957
1957 Scott
1970 de Jongh
Łukasiewicz 1952
Kreisel and Putnam 1957
Scott 1957
1970 de Jongh
Citkin 1979
1984 Visser
Skura Rozière 1989
1992
Iemhoff 2001b
2014
G and Iemhoff
Overview
Overview
Overview

Characterisation of $BB_n$
Overview

Internalising Non-derivability

Characterisation of $BB_n$
Internalising Non-derivability
Theorem (G 2014):
A formula $A$ is derivable iff $\sigma A \vdash \Delta$
yields a classically derivable $C \in \Delta$, for all $\sigma$ and $\Delta$. 

Theorem (G 2014): A formula $A$ is derivable iff $\sigma A \vdash \Delta$ yields a classically derivable $C \in \Delta$, for all $\sigma$ and $\Delta$.

Suppose $\vdash_{IPC} A$ and $\sigma A \vdash \Delta$. 
Theorem (G 2014):
A formula $A$ is derivable iff $\sigma A \vdash \Delta$
yields a classically derivable $C \in \Delta$, for all $\sigma$ and $\Delta$.

Suppose $\vdash_{IPC} A$ and $\sigma A \vdash \Delta$. It follows that $\vdash \sigma A$, so there is a
$C \in \Delta$ with $\vdash_{IPC} C$. 
Theorem (G 2014): A formula $A$ is derivable iff $\sigma A \vdash \Delta$ yields a classically derivable $C \in \Delta$, for all $\sigma$ and $\Delta$.

Suppose $\vdash_{IPC} A$ and $\sigma A \vdash \Delta$. It follows that $\vdash \sigma A$, so there is a $C \in \Delta$ with $\vdash_{IPC} C$. Hence $\vdash_{CPC} C$, as desired.
Corollary:
There is no proper extension of IPC that inherits all its admissible rules.
Corollary:
There is no proper extension of IPC that inherits all its admissible rules.

Suppose \( L \supsetneq IPC \).
Corollary:
There is no proper extension of IPC that inherits all its admissible rules.

Suppose $L \supsetneq IPC$. This gives a $A \in L - IPC$. 
Corollary:
There is no proper extension of IPC that inherits all its admissible rules.

Suppose $L \supsetneq \text{IPC}$. This gives a $A \in L - \text{IPC}$. Hence there is a $\sigma$ and a $\Delta$ with $\not\vdash_{\text{CPC}} C$ for all $C \in \Delta$ such that

$$\sigma A \vdash_{\text{IPC}} \Delta.$$
Corollary:
There is no proper extension of IPC that inherits all its admissible rules.

Suppose $L \supsetneq IPC$. This gives a $A \in L - IPC$. Hence there is a $\sigma$ and a $\Delta$ with $\not\vdash_{CPC} C$ for all $C \in \Delta$ such that

$$\sigma A \vdash_L \Delta.$$
Corollary:
There is no proper extension of IPC that inherits all its admissible rules.

Suppose $L \supsetneq IPC$. This gives a $A \in L - IPC$. Hence there is a $\sigma$ and a $\Delta$ with $\not\vdash_{CPC} C$ for all $C \in \Delta$ such that

$$\sigma A \vdash_L \Delta.$$ 

But $\vdash_L A$, so $\vdash_L \sigma A$ holds as well.
Corollary:
There is no proper extension of IPC that inherits all its admissible rules.

Suppose $L \supsetneq IPC$. This gives a $A \in L - IPC$. Hence there is a $\sigma$ and a $\Delta$ with $\not\vdash_{CPC} C$ for all $C \in \Delta$ such that

$$\sigma A \vdash_{L} \Delta.$$ 

But $\vdash_{L} A$, so $\vdash_{L} \sigma A$ holds as well. This yields some $C \in \Delta$ with such that $\vdash_{L} C$. 


Corollary:
There is no proper extension of IPC that inherits all its admissible rules.

Suppose $L \supsetneq IPC$. This gives a $A \in L - IPC$. Hence there is a $\sigma$ and a $\Delta$ with $\not\vdash_{CPC} C$ for all $C \in \Delta$ such that

$$\sigma A \not\vdash_L \Delta.$$ 

But $\vdash_L A$, so $\vdash_L \sigma A$ holds as well. This yields some $C \in \Delta$ with such that $\vdash_{CPC} C$, a contradiction.
Corollary (Iemhoff, 2001a): IPC is the maximal intermediate logic with the rules below, for all $n$.

$$\left( \bigvee_{i=1}^{n} C_i \rightarrow A \right) \rightarrow \bigvee_{j=1}^{n} C_j$$

$$\left\{ \left( \bigvee_{i=1}^{n} C_i \rightarrow A \right) \rightarrow \bigvee C_j \right\}_{j=1}^{n}$$
The universal model is the "smallest" model into which every finite model fits.
The universal model $U(X)$ is the “smallest” model on $X$ into which every finite model on $X$ fits.
The universal model is complete.
The universal model is complete: $U(X) \models A$ iff $\vdash A$ for all $A \in \mathcal{L}(X)$.
up $k$
Expressing Extensions
Extension Property
Extension Property
$n^{th}$ Extension Property
Visser Rules

\[
\left( \bigvee_{i=1}^{n} C_i \rightarrow A \right) \rightarrow \bigvee_{j=1}^{n} C_j
\]

\[
\begin{array}{c}
\left( \bigvee_{i=1}^{n} C_i \rightarrow A \right) \rightarrow C_j \\
\{ \left( \bigvee_{i=1}^{n} C_i \rightarrow A \right) \rightarrow C_j \}_{j=1}^{n}
\end{array}
\]
\[ (n \lor i = 1) \land (w_i \neq 0) \lor (n_i = 1) \land (w_i) \]
semantics

syntax
semantics

syntax

\[
\forall i = 1 \text{ and } w_i ! \lor \forall i = 1 \text{ and } w_i
\]
semantics : : syntax

\[ w_1 \quad \ldots \quad w_n \]
\[
\left\{ \bigvee_{\mathbf{p} \in \mathbf{w}_1} \left( \bigwedge_{\mathbf{m} \in \mathbf{w}_1} \mathbf{I} = \mathbf{I} \wedge \mathbf{w}_1 \mathbf{up} \mathbf{w}_1 \mathbf{down} \mathbf{w}_1 \right) \right\} \left( \bigvee_{n=1}^{n} \mathbf{w}_n \right)
\]
\[
\left( \bigvee_{i=1}^{n} \text{nd } w_i \rightarrow \bigvee_{i=1}^{n} \text{up } w_i \right) \rightarrow \bigvee_{i=1}^{n} \text{nd } w_i
\]

\[
\left\{ \left( \bigvee_{i=1}^{n} \text{nd } w_i \rightarrow \bigvee_{i=1}^{n} \text{up } w_i \right) \rightarrow \text{nd } w_j \right\}_{j=1}^{n}
\]
\[
\left( \bigvee_{i=1}^{n} \text{nd } w_i \rightarrow \bigvee_{i=1}^{n} \text{up } w_i \right) \rightarrow \bigvee_{i=1}^{n} \text{nd } w_i
\]

\[
\left\{ \left( \bigvee_{i=1}^{n} \text{nd } w_i \rightarrow \bigvee_{i=1}^{n} \text{up } w_i \right) \rightarrow \text{nd } w_j \right\}_{j=1}^{n}
\]
\[
\left( \bigvee_{i=1}^{n} C_i \rightarrow A \right) \rightarrow \bigvee_{i=1}^{n} C_j
\]

\[
\left\{ \left( \bigvee_{i=1}^{n} C_i \rightarrow A \right) \rightarrow C_j \right\}_{j=1}^{n}
\]
A logic with the finite model property admits the Visser rules iff it has the extension property.
A logic with the finite model property admits the Visser rules up to $n$ iff it has the extension property up to $n$. 
Characterisation of $BB_n$
$$BB_n = IPC + \bigwedge_{i=0}^{n} \left( \left( x_i \rightarrow \bigvee_{j \neq i} x_j \right) \rightarrow \bigvee_{j \neq i} x_j \right) \rightarrow \bigvee_{i=0}^{n} x_i$$
\(\text{BB}_n \not
iff \text{ there is a finite, proper and at most } n\text{-fold branching tree } T \text{ with } T \not
iff \ A.\)
Theorem (G 2014):
If $T$ is a finite, proper, and at most $n$-fold branching tree, then
$	ext{nd } m_T \sim \{\text{nd } w \mid w \in T \text{ maximal}\}$. 
Theorem (G 2014):
A formula \( A \) is derivable in BB\(_n\) iff \( \sigma A \vdash \Delta \) yields a classically derivable \( C \in \Delta \), for all \( \sigma \) and \( \Delta \).
Theorem (G 2014): A formula $A$ is derivable in $\text{BB}_n$ iff $\sigma A \vdash \Delta$ yields a classically derivable $C \in \Delta$, for all $\sigma$ and $\Delta$.

Suppose $\not\vdash_{\text{BB}_n} A$. 
Theorem (G 2014):
A formula $A$ is derivable in $\text{BB}_n$ iff
$\sigma A \vdash \Delta$ yields a classically derivable
$C \in \Delta$, for all $\sigma$ and $\Delta$.

Suppose $\not\vdash_{\text{BB}_n} A$. Then there is a finite, proper, and at most
$n$-fold branching tree $T$ such that $T \not\models A$. 
Theorem (G 2014):
A formula $A$ is derivable in $\text{BB}_n$ iff
$\sigma A \vdash \Delta$ yields a classically derivable $C \in \Delta$, for all $\sigma$ and $\Delta$.

Suppose $\not\vdash_{\text{BB}_n} A$. Then there is a finite, proper, and at most $n$-fold branching tree $T$ such that $T \not\models A$. There is a $\sigma$ such that

$\sigma A \vdash_{\text{BB}_n} \text{nd } m_T$
Theorem (G 2014): A formula $A$ is derivable in $\mathbb{BB}_n$ iff $\sigma A \vdash \Delta$ yields a classically derivable $C \in \Delta$, for all $\sigma$ and $\Delta$.

Suppose $\not\vdash_{\mathbb{BB}_n} A$. Then there is a finite, proper, and at most $n$-fold branching tree $T$ such that $T \not\models A$. There is a $\sigma$ such that

$$\sigma A \vdash_{\mathbb{BB}_n} \text{nd } m_T \models \{ \text{nd } w \mid w \in T \text{ maximal} \}.$$
Corollary (G 2014): BB\(_n\) is the maximal intermediate logic with the rules below.

\[
(\bigvee_{i=1}^{n} C_i \rightarrow A) \rightarrow \bigvee_{j=1}^{n} C_j
\]

\[
\{(\bigvee_{i=1}^{n} C_i \rightarrow A) \rightarrow \bigvee C_j\}_{j=1}^{n}
\]


